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On the time derivatives of Boltzmann's H function

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Abstract. It is proved that the derivatives with respect to time of Boltzmann's H function alternate in sign for a gas whose behaviour may be described by the linearized Boltzmann equation.

The question of what is the particular property of Boltzmann's H function which singles it out from a wide class of functionals of the Boltzmann solution has been considered by McKean (1966). He has suggested that such a property may be that the successive derivatives of the H function with respect to time alternate in sign; that is, $(-)^n \mathrm{d}^n H/\mathrm{d}t^n \geqslant 0$ for all integral n. It is clear that such a condition implies that $|\mathrm{d}^n H/\mathrm{d}t^n|$ is a monotonically decreasing function of t for all integral n, and this is a sufficient condition for H to tend to a constant minimum value as t tends to infinity. The question of whether the successive time derivatives of H do, in fact, alternate in sign has been recently discussed by Harris (1967). He succeeded in proving the property to hold for the case of a discrete-velocity gas, but it appears that he was unable to extend the result to a real gas. It is the purpose of the present paper to do this for the case of a dilute gas whose behaviour can be described by the linearized Boltzmann equation. The situation in the case of dense gases is more complicated and will not be considered further here.

If $f(\boldsymbol{v})$ is the distribution function for molecules of velocity $\boldsymbol{v}$, Boltzmann's H function is defined by

$$H = \int f \ln f \, \mathbf{d}\boldsymbol{v}. \tag{1}$$

We let

$$f = f^0(1+\phi) \tag{2}$$

where f^0 is the Maxwell equilibrium distribution function for a gas with density and energy per unit volume equal to that of the given gas. This yields

$$H = \int \{f^0 \ln f^0 + f^0(1+\ln f^0)\phi + \tfrac{1}{2} f^0 \phi^2\} \, \mathbf{d}\boldsymbol{v} \tag{3}$$

on expanding $\ln(1+\phi)$ and retaining terms up to ϕ^2 in the integrand of equation (1). The contribution to the right-hand side of equation (3) arising from the terms linear in ϕ vanishes owing to the equality of density and energy for the gases described by f and f^0, and we obtain

$$H = (1, \ln f^0) + \tfrac{1}{2}(\phi, \phi) \tag{4}$$

where the scalar product of any two functions $A(\boldsymbol{v})$ and $B(\boldsymbol{v})$ is defined by

$$(A, B) = \int f^0 A(\boldsymbol{v}) B(\boldsymbol{v}) \, \mathbf{d}\boldsymbol{v}.$$

It then follows from equation (4) that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{2}\left\{\left(\phi, \frac{\partial \phi}{\partial t}\right) + \left(\frac{\partial \phi}{\partial t}, \phi\right)\right\}. \tag{5}$$

Now it is proved by Chapman and Cowling (1952) that with the present notation

$$\frac{\partial \phi}{\partial t} = L\phi \tag{6}$$

where L is a real linear integral operator. Further, Chapman and Cowling show that for any two functions θ and ψ

$$(\theta, L\psi) = (L\theta, \psi) \quad \text{and} \quad (\theta, L\theta) \leqslant 0. \tag{7}$$

From the first of these relations, together with equations (5) and (6), it follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = (\phi, L\phi).$$

A further differentiation of this equation gives

$$\begin{aligned}\frac{\mathrm{d}^2H}{\mathrm{d}t^2} &= \left(\frac{\partial\phi}{\partial t}, L\phi\right) + \left(\phi, L\frac{\partial\phi}{\partial t}\right) \\ &= 2(\phi, L^2\phi)\end{aligned}$$

and repeating this process yields

$$\frac{\mathrm{d}^nH}{\mathrm{d}t^n} = 2^{n-1}(\phi, L^n\phi) \qquad (1 \leqslant n \leqslant \infty). \tag{8}$$

Now it is readily shown from the relations (7) that

$$(-)^n(\theta, L^n\theta) \geqslant 0$$

for all integral n and hence it follows from equation (8) that $(-)^n \, \mathrm{d}^nH/\mathrm{d}t^n \geqslant 0$.

An alternative proof of this result, which gives further insight into the time variation of H, can be formulated by expanding ϕ in terms of the orthonormal eigenfunctions θ_p of L, which we shall assume form a complete set. We let

$$\phi(\boldsymbol{v}, t) = \sum_p C_p(t)\theta_p(\boldsymbol{v}) \tag{9}$$

which, substituted into equation (6), gives

$$\frac{\mathrm{d}C_p}{\mathrm{d}t} = \lambda_p C_p \tag{10}$$

where $L\theta_p = \lambda_p\theta_p$. If at a particular time, say $t = 0$, ϕ takes the value Γ, we then obtain

$$\phi = \sum_p (\theta_p, \Gamma) \exp(\lambda_p t)\theta_p.$$

On substituting this into equation (4) we find an explicit form for H as a function of time:

$$H = (1, \ln f^0) + \tfrac{1}{2} \sum_p (\theta_p, \Gamma)^2 \exp(2\lambda_p t)$$

and

$$\frac{\mathrm{d}^nH}{\mathrm{d}t^n} = 2^{n-1} \sum_p \lambda_p{}^n(\theta_p, \Gamma)^2 \exp(2\lambda_p t). \tag{11}$$

It follows from the relations (7) that $\lambda_p \leqslant 0$. Thus the right-hand side of equation (11) is positive for n even and negative for n odd which is the required result.

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